Exercise 2.4

Question 1:

Verify that the numbers given alongside of the cubic polynomials below are their zeroes. Also verify the relationship between the zeroes and the coefficients in each case:

(i). $2x^3 + x^2 - 5x + 2;\ \frac{1}{2}, 1, -2$ **(ii).** $x^3 - 4x^2 + 5x - 2;\quad 2, 1, 1$

Answer 1:

(i) $2x^3 + x^2 - 5x + 2;\ \frac{1}{2}, 1, -2$

Let $p(x) = 2x^3 + x^2 - 5x + 2$

Therefore , $p\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^2 - 5\left(\frac{1}{2}\right) + 2$

$= 2 \times \frac{1}{8} + \frac{1}{4} - 5 \times \frac{1}{2} + 2$

$= \frac{2}{4} - \frac{5}{2} + 2$

$= \frac{5}{2} - \frac{5}{2} = 0$

So, $\frac{1}{2}$ is one of the zeroes of $p(x)$.

Now, $p(1) = 2(1)^3 + (1)^2 - 5(1) + 2$

$= 2 \times 1 + 1 - 5 \times 1 + 2$

$= 3 - 5 + 2$

$= 5 - 5 = 0$

So, 1 is also the zero of $p(x)$.

Now, $p(-2) = 2(-2)^3 + (-2)^2 - 5(-2) + 2$

$= 2 \times (-8) + 4 + 10 + 2$

$= -16 + 4 + 10 + 2$

$= -16 + 16 = 0$

So, -2 is also the zero of $p(x)$.

Therefore, $\frac{1}{2}$, 1 and -2 are the zeroes of $p(x)$.

Now, let $\alpha = \frac{1}{2}, \beta = 1$ and $\gamma = -2$

$$\alpha + \beta + \gamma = \frac{1}{2} + 1 + (-2) = \frac{3-4}{2} = \frac{-1}{2} = \frac{-(1)}{2} = \frac{-(\text{Cofficient of } x^2)}{\text{Cofficient of } x^3}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{1}{2} \times 1 + 1 \times (-2) + (-2) \times \frac{1}{2} = \frac{-5}{2} = \frac{-5}{2} = \frac{\text{Cofficient of } x}{\text{Cofficient of } x^3}$$

$$\alpha\beta\gamma = \frac{1}{2} \times 1 \times (-2) = -1 = \frac{-2}{2} = \frac{-(\text{Constant term})}{\text{Cofficient of } x^3}$$

Hence, the relation between zeroes and coefficients is verified.

(ii) $x^3 - 4x^2 + 5x - 2;\quad 2, 1, 1$

Let, $p(x) = x^3 - 4x^2 + 5x - 2$

Therefore , $p(2) = (2)^3 - 4(2)^2 + 5(2) - 2$

$= 8 - 16 + 10 - 2$

$= 18 - 18 = 0$

So, 2 is one of the zeroes of $p(x)$.

Now, $p(1) = (1)^3 - 4(1)^2 + 5(1) - 2$

$= 1 - 4 + 5 - 2$

$= 6 - 6 = 0$

So, 1 is also the zero of $p(x)$.

Therefore, 2, 1 and 1 are the zeroes of $p(x)$.

Now, let $\alpha = 2, \beta = 1$ and $\gamma = 1$

$$\alpha + \beta + \gamma = 2 + 1 + 1 = 4 = \frac{-(-1)}{1} = \frac{-(\text{Cofficient of } x^2)}{\text{Cofficient of } x^3}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = 2 \times 1 + 1 \times 1 + 1 \times 2 = 2 + 1 + 2 = 5 = \frac{5}{1} = \frac{\text{Cofficient of } x}{\text{Cofficient of } x^3}$$

$$\alpha\beta\gamma = 2 \times 1 \times 1 = 2 = \frac{-(-2)}{1} = \frac{-(\text{Constant term})}{\text{Cofficient of } x^3}$$

Hence, the relation between zeroes and coefficients is verified.

Question 2:

Find a cubic polynomial with the sum, sum of the product of its zeroes taken two at a time, and the product of its zeroes as 2, –7, –14 respectively.

Answer 2:

Let, $p(x) = ax^3 + bx^2 + cx + d$ be a cubic polynomial whose zeroes are α, β and γ.
Given that:
$\alpha + \beta + \gamma = 2$
$\alpha\beta + \beta\gamma + \gamma\alpha = -7$
$\alpha\beta\gamma = -14$

We know that,

$$\alpha + \beta + \gamma = \frac{-(\text{Cofficient of } x^2)}{\text{Cofficient of } x^3}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{\text{Cofficient of } x}{\text{Cofficient of } x^3}$$

$$\alpha\beta\gamma = \frac{-(\text{Constant term})}{\text{Cofficient of } x^3}$$

Therefore,

$$\alpha + \beta + \gamma = \frac{-(\text{Cofficient of } x^2)}{\text{Cofficient of } x^3} = \frac{-b}{a} = \frac{2}{1}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{\text{Cofficient of } x}{\text{Cofficient of } x^3} = \frac{c}{a} = \frac{-7}{1}$$

$$\alpha\beta\gamma = \frac{-(\text{Constant term})}{\text{Cofficient of } x^3} = \frac{-d}{a} = \frac{-14}{1}$$

On comparing, $a = 1, b = -2, c = -7$ and $d = 14$

Hence, the required cubic polynomial is $p(x) = x^3 - 2x^2 - 7x + 14$.

Question 3:

If the zeroes of the polynomial $x^3 - 3x^2 + x + 1$ are $a - b$, a, $a + b$, find a and b.

Answer 3:

We know that,

$$\text{Sum of zeroes} = \frac{-(\text{Cofficient of } x^2)}{\text{Cofficient of } x^3}$$

Therefore,

$$(\mathrm{a} - \mathrm{b}) + \mathrm{a} + (\mathrm{a} + \mathrm{b}) = \frac{-(-3)}{1}$$

$\Rightarrow 3a = 3$
$\Rightarrow a = 1$

$$\text{Product of zeroes} = \frac{-(\text{Constant term})}{\text{Cofficient of } x^3}$$

Therefore,

$$(\mathrm{a} - \mathrm{b})(\mathrm{a})(\mathrm{a} + \mathrm{b}) = \frac{-(1)}{1}$$

$\Rightarrow (1 - \mathrm{b})1(1 + \mathrm{b}) = -1$ [Because $a = 1$]
$\Rightarrow 1 - b^2 = -1 \quad \Rightarrow b^2 = 2 \quad \Rightarrow b = \pm\sqrt{2}$
Hence, $a = 1$ and $b = \pm\sqrt{2}$

Question 4:

If two zeroes of the polynomial $x^4 - 6x^3 - 26x^2 + 13x - 35$ are $2 \pm \sqrt{3}$, find other zeroes.

Answer 4:

Let $p(x) = x^4 - 6x^3 - 26x^2 + 13x - 35$
Given that: $2 + \sqrt{3}$ and $2 - \sqrt{3}$ are two of the zeroes of $p(x)$.
So, $(x - 2 - \sqrt{3})$ and $(x - 2 + \sqrt{3})$ are the factors of $p(x)$.
Therefore, $(x - 2)^2 - (\sqrt{3})^2$ is a factor of $p(x)$.
$\Rightarrow x^2 - 4x + 1$ is a factor of $p(x)$.
[As $(x - 2)^2 - (\sqrt{3})^2 = x^2 - 4x + 1$]

$$\begin{array}{r|l} & x^2 - 2x - 35 \\ \hline x^2 - 4x + 1 & x^4 - 6x^3 - 26x^2 + 138x - 35 \\ & x^4 - 4x^3 + x^2 \\ & - \quad + \quad - \\ \cline{2-2} & -2x^3 - 27x^2 + 138x - 35 \\ & -2x^3 + 8x^2 - 2x \\ & + \quad - \quad + \\ \cline{2-2} & -35x^2 + 140x - 35 \\ & -35x^2 + 140x - 35 \\ & + \quad - \quad + \\ \cline{2-2} & 0 \end{array}$$

Therefore,
$p(x) = x^4 - 6x^3 - 26x^2 + 13x - 35$
$= (x^2 - 4x + 1)(x^2 - 2x - 35)$
$= (x^2 - 4x + 1)(x^2 - 7x + 5x - 35)$
$= (x^2 - 4x + 1)[x(x - 7) + 5(x - 7)]$
$= (x^2 - 4x + 1)(x + 5)(x - 7)$
To get the other zeroes, putting $x + 5 = 0$ and $x - 7 = 0$, we get

$x = -5$ and $x = 7$

Hence, the other two zeroes of $p(x)$ are $x = -5$ and $x = 7$.

Question 5:

If the polynomial $x^4 - 6x^3 + 16x^2 - 25x + 10$ is divided by another polynomial $x^2 - 2x + k$, the remainder comes out to be $x + a$, find k and a.

Answer 5:

Given that:
Divisor $= x^2 - 2x + k$
Dividend $= x^4 - 6x^3 + 16x^2 - 25x + 10$
Remainder $= x + a$

We know that:

Dividend = Divisor × Quotient + Remainder

Therefore,

$$x^4 - 6x^3 + 16x^2 - 25x + 10 = (x^2 - 2x + k) \times \text{Quotient} + (x + a)$$

$$x^4 - 6x^3 + 16x^2 - 25x + 10 - (x + a) = (x^2 - 2x + k) \times \text{Quotient}$$

$$\Rightarrow \frac{x^4 - 6x^3 + 16x^2 - 26x + 10 - a}{x^2 - 2x + k} = \text{Quotient}$$

So, if the polynomial $x^4 - 6x^3 + 16x^2 - 26x + 10 - a$ is divided by $x^2 - 2x + k$, the remainder will be zero.

$$\begin{array}{r|l}
 & \qquad\qquad\quad x^2-4x+(8-k) \\ \hline
x^2-2x+k & x^4-6x^3+16x^2-26x+10-a \\
 & x^4-2x^3+\ kx^2 \\
 & -\ \ +\ \ \ - \\ \cline{2-2}
 & \quad -4x^3+(16-k)x^2-26x \\
 & \quad -4x^3+\quad 8x^2-4kx \\
 & \quad +\ \ \ -\qquad\ \ + \\ \cline{2-2}
 & \qquad\quad (8-k)x^2-(26-4k)x+10-a \\
 & \qquad\quad (8-k)x^2-(16-2k)x+(8k-k^2) \\
 & \qquad\qquad -\qquad\ \ +\qquad\ \ - \\ \cline{2-2}
 & \qquad\qquad\qquad (-10+2k)x+(10-a-8k+k^2)
\end{array}$$

On comparing,
$-10 + 2\text{k} = 0$
$\Rightarrow 2k = 10$
$\Rightarrow k = 5$
and
$10 - a - 8k + k^2 = 0$
$\Rightarrow 10 - a - 8 \times 5 + 5^2 = 0$ [As $k = 5$]
$\Rightarrow 10 - a - 40 + 25 = 0$
$\Rightarrow -a - 5 = 0$
$\Rightarrow a = -5$
Hence, $k = 5$ and $a = -5$