Exercise 1.3

Question 1:
Prove that $\sqrt{5}$ is irrational.

Answer 1:
Let $\sqrt{5}$ is a rational number.

Therefore, we can find two integers a, b $(b \neq 0)$ such that $\sqrt{5} = \frac{a}{b}$ Let a and b have a common factor other than 1. Then we can divide them by the common factor, and assume that a and b are co-prime.

$$a = \sqrt{5}b$$
$$\Rightarrow a^2 = 5b^2$$

Therefore, a^2 is divisible by 5 and it can be said that a is divisible by 5.

Let $a = 5k$, where k is an integer

$$(5k)^2 = 5b^2$$
$$\Rightarrow 5k^2 = b^2$$

This means that b^2 is divisible by 5 and hence, b is divisible by 5.

This implies that a and b have 5 as a common factor.

And this is a contradiction to the fact that a and b are co-prime.

Hence, $\sqrt{5}$ *cannot* be expressed as $\frac{p}{q}$ or it can be said that $\sqrt{5}$ is irrational.

Question 2:
Prove that $3 + 2\sqrt{5}$ is irrational.

Answer 2:
Let $3 + 2\sqrt{5}$ is rational.

Therefore, we can find two co-prime integers a, b $(b \neq 0)$ such that

$$3 + 2\sqrt{5} = \frac{a}{b}$$
$$\Rightarrow 2\sqrt{5} = \frac{a}{b} - 3$$
$$\Rightarrow \sqrt{5} = \frac{1}{2}\left(\frac{a}{b} - 3\right)$$

Since a and b are integers, $\frac{1}{2}\left(\frac{a}{b} - 3\right)$ will also be rational and therefore, $\sqrt{5}$ is rational.

This contradicts the fact that $\sqrt{5}$ is irrational. Hence, our assumption that $3 + 2\sqrt{5}$ is rational is false. Therefore, $3 + 2\sqrt{5}$ is irrational.

Question 3:
Prove that the following are irrationals:

(i) $\frac{1}{\sqrt{2}}$ **(ii)** $7\sqrt{5}$ **(iii)** $6 + \sqrt{2}$

Answer 3:
(i) $\frac{1}{\sqrt{2}}$

Let $\frac{1}{\sqrt{2}}$ is rational.

Therefore, we can find two co-prime integers a, b $(b \neq 0)$ such that

$$\frac{1}{\sqrt{2}} = \frac{a}{b}$$

Or

$$\sqrt{2} = \frac{b}{a}$$

$\frac{b}{a}$ is rational as a and b are integers.

Therefore, $\sqrt{2}$ is rational which contradicts to the fact that $\sqrt{2}$ is irrational.

Hence, our assumption is false and $\frac{1}{\sqrt{2}}$ is irrational.

(ii) $7\sqrt{5}$

Let $7\sqrt{5}$ is rational.

Therefore, we can find two co-prime integers a, b $(b \neq 0)$ such that

$$7\sqrt{5} = \frac{a}{b}$$

$$\Rightarrow \sqrt{5} = \frac{a}{7b}$$

$\frac{a}{7b}$ is rational as a and b are integers.

Therefore, $\sqrt{5}$ should be rational.

This contradicts the fact that $\sqrt{5}$ is irrational. Therefore, our assumption that $7\sqrt{5}$ is rational is false. Hence, $7\sqrt{5}$ is irrational.

(iii) $6 + \sqrt{2}$

Let $6 + \sqrt{2}$ be rational.

Therefore, we can find two co-prime integers a, b $(b \neq 0)$ such that

$$6 + \sqrt{2} = \frac{a}{b}$$

$$\Rightarrow \sqrt{2} = \frac{a}{b} - 6$$

Since a and b are integers, $\frac{a}{b} - 6$ is also rational and hence, $\sqrt{2}$ should be rational. This contradicts the fact that $\sqrt{2}$ is irrational. Therefore, our assumption is false and hence, $6 + \sqrt{2}$ is irrational.